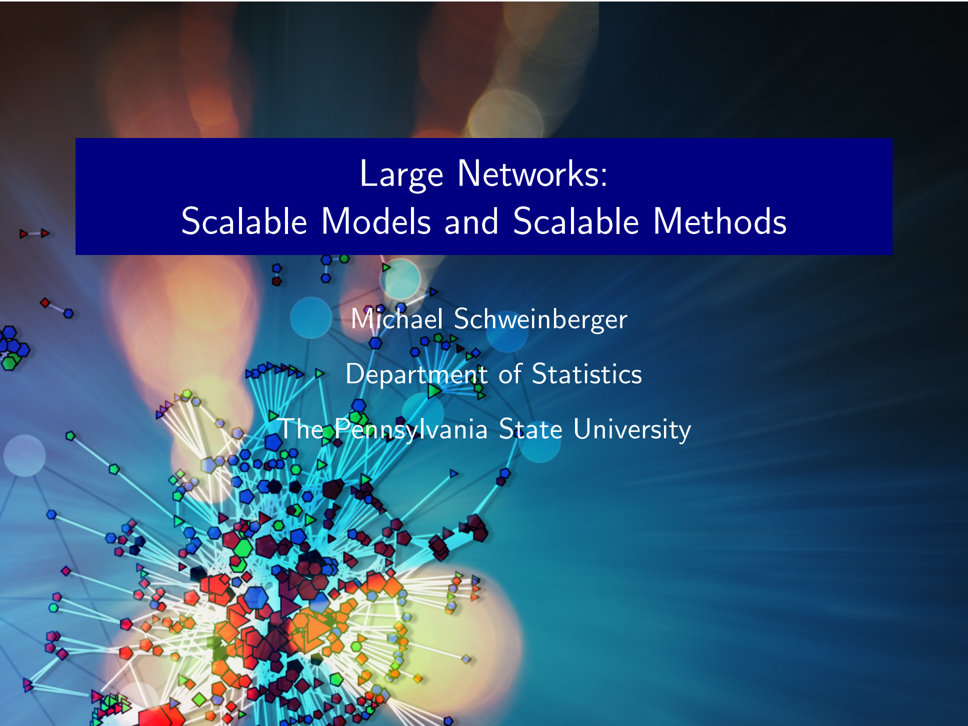

Large Networks: Scalable Models and Scalable Methods

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Outline

Motivation

Scalable models

- Advantage: theoretical guarantees
- Advantage: scalable methods and software

Scalable methods and software

Structure

Motivation

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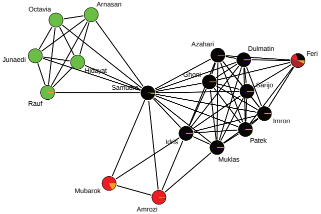
Scalable methods and software

Before the age of data science: small networks with dozens of members (e.g., terrorist network behind Bali bombing in 2002):



Age of data science: large networks with thousands or millions of members (e.g., hate speech on Twitter (X)):

What do we know about large networks?

- Large populations are **heterogeneous.**
- **Some population members are closer than others.**
- Large population networks are **sparse.**
- **Connections** depend on other connections, but **do not depend on all other connections.**

In large populations, populations members are not aware of the attributes and connections of most other population members and hence cannot be directly affected by them.

How can we study large networks?

To study large networks, we need

- scalable models
- scalable methods and software.

The bulk of the literature focuses on *scalable methods and software,* while ignoring the importance of *scalable models.*

The importance of scalable models

Lessons from model degeneracy and other undesirable properties of classic models (e.g., Frank and Strauss 1986):

- **Model degeneracy is rooted in a lack of structure:** Without additional structure, it is challenging to build scalable models that are well-behaved in small and large networks.
- **Additional structure** is important in practice and theory, and **helps construct scalable models.**
- **The study of large networks requires scalable models, in addition to scalable methods and software.**

Schweinberger (Journal of the American Statistical Association, Theory & Methods, 2011)

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Scalable methods and software

Notation

Define

$$X_{i,j} \quad := \quad \begin{cases} 1 & \text{if } i \text{ and } j \text{ are connected} \\ 0 & \text{otherwise} \end{cases}$$

Extensions to directed and weighted networks are possible.

Scalable models

Build scalable models that respect the fundamental features of large networks, including but not limited to:

- Large populations are **heterogeneous.**
- **Some population members are closer than others.**
- Large population networks are **sparse.**
- **Connections depend on subsets of other connections.**

Scalable models with local dependence

Scalable models with local dependence

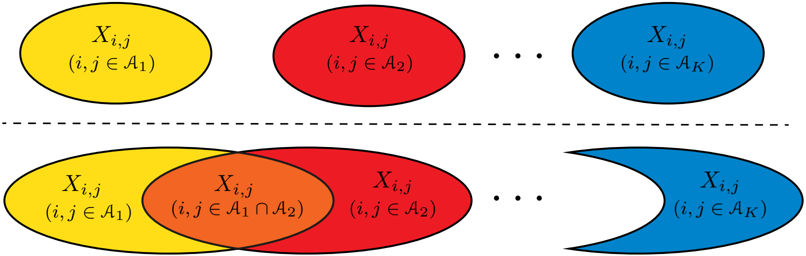
A probability law $\mathbb{P}$ governing connection indicators $X_{i,j}$ induces local dependence if, for each pair of members $\{i, j\}$, the conditional probability of the event $\{X_{i,j} = 1\}$ depends on the connections among a subset of other members:

$$\mathbb{P}(X_{i,j} = 1 \mid \text{universe}) \quad = \quad \mathbb{P}(X_{i,j} = 1 \mid \text{subset of universe})$$

Schweinberger and Handcock (Journal of the Royal Statistical Society, Series B, 2015)

Scalable models with local dependence

The construction of models with local dependence is facilitated by **additional structure,** e.g., $K \geq 2$ subpopulations $\mathcal{A}_1, \ldots, \mathcal{A}_K$:



- Observed: e.g., multilevel networks
- Unobserved: learned from data.

Scalable models with local dependence

Scalable models with local dependence leveraging additional structure

A probability law $\mathbb{P}$ governing connection indicators $X_{i,j}$ in a population consisting of $K \geq 2$ non-overlapping subpopulations $\mathcal{A}_1, \ldots, \mathcal{A}_K$ induces local dependence if

$$\mathbb{P}(\boldsymbol{X}) \;=\; \prod_{k=1}^{K} \mathbb{P}_{k,k}(\boldsymbol{X}_{\mathcal{A}_k,\mathcal{A}_k}) \prod_{l=k+1}^{K} \mathbb{P}_{k,l}(\boldsymbol{X}_{\mathcal{A}_k,\mathcal{A}_l})$$

where the dependence is restricted to connections within the same subpopulation.

The building blocks $\mathbb{P}_{k,k}$ and $\mathbb{P}_{k,l}$ can be parameterized by generalizations of logistic regression models (ERGMs), which have game-theoretic foundations: see Butts (2009) and Mele (2017).

More exciting developments

More exciting developments down the road:

- Models with overlapping subpopulations ("overlapping social circles"): Stewart and Schweinberger (Annals of Statistics, invited revision, 2023)
- Models for studying relationships among attributes under network interference in large populations: Fritz, Schweinberger, Bhadra, and Hunter (forthcoming).

Structure

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Scalable methods and software

Advantage: theoretical guarantees

Challenges:

- Network data are **non-standard data.**
- Network data are **dependent data.**
- **It may not be possible to observe independent replications from the same source,** so that classical LLNs and CLTs cannot be invoked to obtain theoretical guarantees.

Existing statistical theory *does not guarantee that estimators of parameters based on dependent network data are close to the truth.*

Advantage: theoretical guarantees

Local dependence facilitates some of the first theoretical guarantees for models with complex dependence:

- Learning non-overlapping subpopulations: Schweinberger (Bernoulli, 2020).
- Learning parameters given non-overlapping subpopulations: Schweinberger and Stewart (Annals of Statistics, 2020).
- Learning parameters (in general): Stewart and Schweinberger (Annals of Statistics, invited revision, 2023)
- Disclaimers: quantifying uncertainty about parameter estimators: Stewart (Bernoulli, invited revision, 2024).

The general mathematical results in Stewart and Schweinberger (2023) underscore the importance of additional structure for the purpose of controlling dependence ($\rightarrow$ local dependence).

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Scalable methods and software

Advantage: scalable methods and software

Local dependence facilitates local computing on subnetworks, which enables large-scale computing:

- Simulation
- Estimation.

Structure

Scalable methods

If the subpopulation structure is unobserved, the model can be estimated in two steps:

- **Step 1:** Estimate the subpopulation structure by taking advantage of the fact that local dependence models are generalizations of stochastic block models and that stochastic block models can be estimated by scalable methods.
- **Step 2:** Estimate the parameters conditional on the estimated subpopulation structure using local computing on subnetwoks.

Babkin, Stewart, Long, and Schweinberger (Computational Statistics & Data Analysis, 2020)

Scalable software

- R package **hergm** (2008–2024), created and maintained by Schweinberger and published in Schweinberger and Luna (Journal of Statistical Software, 2018).
- R package **lighthergm** based on **hergm** (2019–present), developed and maintained by Sansan Inc. with the help of Mele and Schweinberger.
- R package **bigergm**, which is the CRAN version of **lighthergm** (2024–present), developed and maintained by Fritz with the help of the `lightergm` and `hergm` core development teams.

All of them implement the scalable methods of Babkin, Stewart, Long, and Schweinberger (2020), but **bigergm** is the most advanced version, and it is the public version on CRAN.

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